Maximal D-avoiding subsets of $\mathbb{Z}$

Nathan Ramesh
Mentor: Christian Gaetz

PRIMES Conference
May 19, 2018

Introduction

- D is a finite set of positive integers
- $S \subset \mathbb{N}$ called *D-avoiding* if there do not exist $x, y \in S$ such that $x - y \in D$

Introduction

- D is a finite set of positive integers
- $S \subset \mathbb{N}$ called *D-avoiding* if there do not exist $x, y \in S$ such that $x - y \in D$

Example

- $\{1, 4, 5, 8, 11\}$ is $\{2, 5\}$-avoiding

Introduction

- D is a finite set of positive integers
- $S \subset \mathbb{N}$ called *D-avoiding* if there do not exist $x, y \in S$ such that $x - y \in D$

Example

- $\{1, 4, 5, 8, 11\}$ is $\{2, 5\}$-avoiding

Example

- $\{1, 3, 5, 7, \cdots\}$ is $\{1, 5\}$-avoiding

Subset Generating Function

- for a subset S, define the generating function

$$S_q = \sum_{n \in S} q^n$$

Subset Generating Function

- for a subset S, define the generating function

$$S_q = \sum_{n \in S} q^n$$

- to compare the size of two subsets A and B, compare the germs of A_q and B_q as $q \to 1^-$, i. e. when $q = 1 - \varepsilon$

Subset Generating Function

- for a subset S, define the generating function

$$S_q = \sum_{n \in S} q^n$$

- to compare the size of two subsets A and B, compare the germs of A_q and B_q as $q \to 1^-$, i. e. when $q = 1 - \varepsilon$
- If $A_q > B_q$ when $q = 1 - \varepsilon$, we write $A \succ B$

Subset Generating Function

- for a subset S, define the generating function

$$S_q = \sum_{n \in S} q^n$$

- to compare the size of two subsets A and B, compare the germs of A_q and B_q as $q \to 1^-$, i. e. when $q = 1 - \varepsilon$
- If $A_q > B_q$ when $q = 1 - \varepsilon$, we write $A \succ B$

Example

- $A = \{0, 2, 4, 6, \cdots\}$
- $B = \{1, 3, 5, 7, \cdots\}$

Subset Generating Function

- for a subset S, define the generating function

$$S_q = \sum_{n \in S} q^n$$

- to compare the size of two subsets A and B, compare the germs of A_q and B_q as $q \to 1^-$, i. e. when $q = 1 - \varepsilon$
- If $A_q > B_q$ when $q = 1 - \varepsilon$, we write $A \succ B$

Example

- $A = \{0, 2, 4, 6, \cdots\}$
- $B = \{1, 3, 5, 7, \cdots\}$

- $A_q = 1 + q^2 + q^4 + \cdots = \frac{1}{1-q^2}$
- $B_q = q + q^3 + q^5 + \cdots = \frac{q}{1-q^2}$

Subset Generating Function

- for a subset S, define the generating function

$$S_q = \sum_{n \in S} q^n$$

- to compare the size of two subsets A and B, compare the germs of A_q and B_q as $q \to 1^-$, i. e. when $q = 1 - \varepsilon$
- If $A_q > B_q$ when $q = 1 - \varepsilon$, we write $A \succ B$

Example

- $A = \{0, 2, 4, 6, \cdots\}$
- $B = \{1, 3, 5, 7, \cdots\}$

- $A_q = 1 + q^2 + q^4 + \cdots = \frac{1}{1-q^2}$
- $B_q = q + q^3 + q^5 + \cdots = \frac{q}{1-q^2}$

- $\implies A \succ B$

Propp's Theorem

Theorem (Propp)

Every germ-maximal D-avoiding subset S of $\mathbb{N}$ is eventually periodic.

Periodicity Implies Rationality

Lemma

Eventual perioicity implies that the associated S_q is a rational function.

Periodicity Implies Rationality

Lemma

Eventual perioicity implies that the associated S_q is a rational function.

Example

For $S = \{0, 1, 3, 4, 6, 7, 9, 10, \cdots\}$,

$$S_q = 1 + q + q^3 + q^4 + q^6 + q^7 + \cdots = \frac{1+q}{1-q^3}$$

Our Extension: Germ Maximality in $\mathbb{Z}$

- generating function workaround:

$$S_q = \sum_{n \in S} q^{|n|}$$

Our Extension: Germ Maximality in $\mathbb{Z}$

- generating function workaround:

$$S_q = \sum_{n \in S} q^{|n|}$$

Theorem (Extension of Propp)

Every germ-maximal D-avoiding subset S of $\mathbb{Z}$ has rational S_q.

Our Extension: Germ Maximality in $\mathbb{Z}$

- generating function workaround:

$$S_q = \sum_{n \in S} q^{|n|}$$

Theorem (Extension of Propp)

Every germ-maximal D-avoiding subset S of $\mathbb{Z}$ has rational S_q.

- Conjectures
  - Any germ-maximal subset of $\mathbb{Z}$ is completely periodic.
    - Not true in $\mathbb{N}$.
    - When $D = \{1, 4, 7\}$,

$$\{0, 1, 3, 6, 9, 15, 18, \cdots\} \succ \{0, 3, 6, 9, 12, 15, 18, \cdots\}$$

  - Any germ-maximal subset of $\mathbb{Z}$ contains 0.

Density

- Density of S defined by

$$\delta(S) = \lim_{n\to\infty} \frac{|S \cap \{0, 1, 2, \cdots, n\}|}{n+1}$$

Density

- Density of S defined by

$$\delta(S) = \lim_{n\to\infty} \frac{|S \cap \{0, 1, 2, \cdots, n\}|}{n+1}$$

- Alternatively

$$\delta(S) = \lim_{q\to 1^-} (1-q) \cdot S_q$$

Density

- Density of S defined by

$$\delta(S) = \lim_{n\to\infty} \frac{|S \cap \{0,1,2,\cdots,n\}|}{n+1}$$

- Alternatively

$$\delta(S) = \lim_{q\to 1^-} (1-q)\cdot S_q$$

Example

The density of $\{0,2,4,6,8,\cdots\}$ is $\frac{1}{2}$.

Maximal Density of D-avoiding set

- Maximal density of a D-avoiding set is defined by

$$\mu(D) = \sup\{\delta(S) : S \text{ is } D\text{-avoiding}\}$$

Maximal Density of D-avoiding set

- Maximal density of a D-avoiding set is defined by

$$\mu(D) = \sup\{\delta(S) : S \text{ is } D\text{-avoiding}\}$$

- Goal: determine μ given D

Lower Bound on μ

Theorem

We have $\mu(D) \geq \frac{1}{|D|+1}$.

Lower Bound on μ

Theorem
We have $\mu(D) \geq \frac{1}{|D|+1}$.

Proof.
Use the following algorithm to greedily build S.

1. Put $0 \in S$.
2. Put all $x + d \in S'$ for all $x \in S$ and $d \in D$.
3. Put the smallest positive integer not currently in S or S' into S. Return to step (2).

$\square$

Lonely Runner Number

- $||x|| = \min(\lceil x \rceil - x, x - \lfloor x \rfloor)$
- $||tD|| = \min_{d \in D} ||td||$

Lonely Runner Number

- $||x|| = \min(\lceil x \rceil - x, x - \lfloor x \rfloor)$
- $||tD|| = \min_{d \in D} ||td||$

- Lonely runner number

$$lr(D) = \sup\{||tD|| : t \in \mathbb{R}\}$$

Lonely Runner Number

- $||x|| = \min(\lceil x \rceil - x, x - \lfloor x \rfloor)$
- $||tD|| = \min_{d \in D} ||td||$
- Lonely runner number

$$lr(D) = \sup\{||tD|| : t \in \mathbb{R}\}$$

Conjecture (Lonely Runner)

The lonely runner conjecture conjectures that $lr(D) \geq \frac{1}{|D|+1}$ for all D.

Connection to Density

- Cantor and Gordon proved that $\mu(D) \geq lr(D)$ for all D.

Connection to Density

- Cantor and Gordon proved that $\mu(D) \geq lr(D)$ for all D.
- Agrees with earlier claim; we have

$$\mu(D) \geq lr(D) \geq \frac{1}{|D|+1}.$$

Connection to Density

- Cantor and Gordon proved that $\mu(D) \geq lr(D)$ for all D.
- Agrees with earlier claim; we have

$$\mu(D) \geq lr(D) \geq \frac{1}{|D|+1}.$$

Conjecture (Harlambis)

For $|D| = 3$, we have $\mu(D) = lr(D)$.

Future Directions

- Explore new special classes of sets D. For example, the cases of finite arithmetic and geometric series have already been completely solved, as well as many classes of three element sets of the form $\{1, j, k\}$.

Future Directions

- Explore new special classes of sets D. For example, the cases of finite arithmetic and geometric series have already been completely solved, as well as many classes of three element sets of the form $\{1, j, k\}$.
- Bounding μ from above in terms of lr or some other value; currently we have no way of even quickly determining a maximal upper bound on the value of μ.

Future Directions

- Explore new special classes of sets D. For example, the cases of finite arithmetic and geometric series have already been completely solved, as well as many classes of three element sets of the form $\{1, j, k\}$.
- Bounding μ from above in terms of lr or some other value; currently we have no way of even quickly determining a maximal upper bound on the value of μ.
- Find out exactly when equality holds in the Theorem and other cases discussed above.

Acknowledgements

We wish to thank:

- Mentor Christian Gaetz
- James Propp
- Dr. Tanya Khovanova, Dr. Slava Gerovitch
- The PRIMES program and the MIT math department